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# Reflection and Polarization of Light by Semi-Infinite Turbid Media: Simple Approximations

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**The paper is devoted to the investigation of the accuracy of the Rozenberg approximation for the reflection function of a semi-infinite absorbing random media. We found that this approximation can be used at $y < 0.5$, where $y = \sqrt{(1-\omega_0)/(3(1-g))}$. Here $\omega_0$ is the single scattering albedo and $g$ is the asymmetry parameter. We also propose the modified approximation, which can be used at least up to $y = 1$–$2$, depending on the observation geometry.** 

*Key Words:* **light reflection; radiative transfer.**

## INTRODUCTION

The task of this paper is to study the reflection function of a semi-infinite turbid medium. This limit can be easily achieved experimentally. For this one should increase the geometrical thickness of a scattering layer or concentration of particles till the measured reflection function reaches its saturation level (1). Thus, the case considered here is of importance for the spectroscopy of turbid media in general and for the spectroscopy of colloids and foams in particular. It is clear that even small absorption in a single scattering event accumulates to a large and easily measured effect due to a random walk of photons in a semi-infinite scattering layer. Thus, the spectral measurements of the reflectance function provide us with the information on absorption coefficient of particles, which can be used, e.g., for the retrieval of their chemical composition.

The solution of the spectroscopic problem is greatly simplified if analytical relations between measured reflection functions and absorption coefficients are used (2, 3). One of such relations has the following form (4–6),

$$R_\infty(\xi, \eta, \varphi) = R^0_\infty(\xi, \eta, \varphi)\exp(-U(\xi, \eta, \varphi)y), \qquad [1]$$

where $R_\infty(\xi, \eta, \varphi)$ is the reflection function of a semi-infinite absorbing turbid layer, $R^0_\infty(\xi, \eta, \varphi)$ is the same function for a nonabsorbing layer with the same microstructure, $\xi = \cos\vartheta_0$, $\eta = \cos\vartheta$, $\varphi$ is the relative azimuthal angle between incident and observed beams, $\vartheta$ is the observation angle, $\vartheta_0$ is the incidence angle,

$$y = 4\sqrt{\frac{1-\omega_0}{3(1-g)}}, \qquad [2]$$

$\omega_0$ is the single scattering albedo, $g$ is the asymmetry parameter,

$$U(\xi, \eta, \varphi) = \frac{K_0(\xi)K_0(\eta)}{R^0_\infty(\xi, \eta, \varphi)}, \qquad [3]$$

$$K_0(\xi) = \frac{3}{2}\int_0^1 R^0_\infty(\xi, \eta)(\xi+\eta)\eta\, d\eta, \qquad [4]$$

$$R^0_\infty(\xi, \eta) = \frac{1}{2\pi}\int_0^{2\pi} R^0_\infty(\xi, \eta, \varphi)\, d\varphi, \qquad [5]$$

The function $K_0(\xi)$ can be approximately calculated with the following formula at $\xi > 0.2$ (7),

$$K_0(\xi) = \frac{3}{7}(1+2\xi) \qquad [6]$$

independently on a scattering medium microstructure.

Equations [l]–[3] and [6] allow us to reduce the calculation of the reflection function of an absorbing medium to that at no absorption. This is an important result, which can be used for various spectroscopic applications (2). Equation [1] is called the Rozenberg equation.

Our task is to study the accuracy of Eq. [1] for different observation geometries and values of $y$. Also we slightly modify Eq. [1] to have a possibility to consider larger values of light absorption in a single scattering event.

## THE MODIFIED ROZENBERG EQUATION

The reflection function is defined as the ratio of the intensity of light reflected from a medium in question to that of an ideally white Lambertian reflector (1). Clearly, this function for a semi-infinite turbid medium decreases with the parameter $\beta = 1 - \omega_0$, which gives us the probability of photon absorption in a single interaction event. This follows from both Eq. [l] and Eq. [2] and



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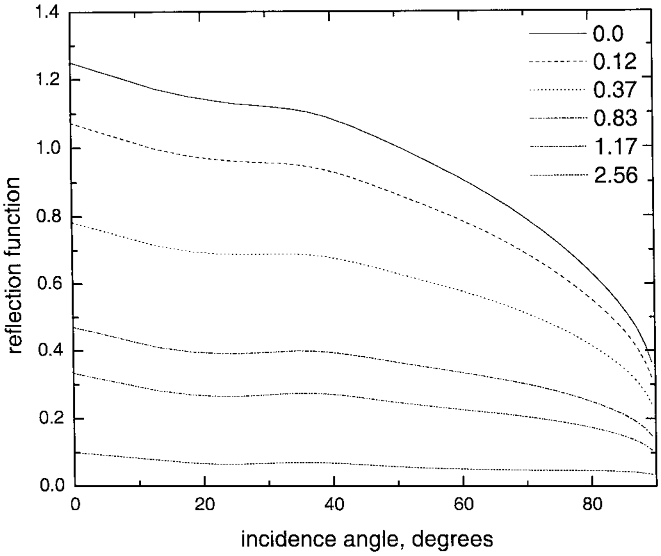


**FIG. 1.** The reflection function $R_\infty(\xi, 1)$ of a semi-infinite cloudy media with droplets characterized by the gamma particle size distribution $f(a) = Aa^6 \exp(-9a/a_{ef})$, where $A = \text{const}$, $a_{ef} = 6$ $\mu$m for different values of $y = 0$, 0.12, 0.37, 0.83, 1.17, and 2.56. It was assumed that the wavelength $\lambda$ is equal to 1.55 $\mu$m. The refractive index of particles $m = 1.3309 - i\chi$, where $\chi = 0.0$, 0.00001, 0.0001, 0.005, 0.001, 0.005.

from the results of the exact solution of the radiative transfer equation, presented in Fig. 1 for different values of $y$. Naturally, it vanishes altogether if the probability of photon absorption, $\beta$ is equal to one. In particular, we see that it is smaller than 0.1 at the nadir observation and $y > 2.56$. The results presented in Fig. 1 are obtained solving the Ambartsumian's nonlinear integral equation for the reflection function of a semi-infinite medium (8).

It should be pointed out that reflection functions for turbid media with the same value of $\beta$, but with different values of $g$ differ. Larger values of $g < 1$ lead to larger $y$ and, therefore, larger reduction of the reflection function $R_\infty$ (see Eq. [1] and Fig. 1). This can be understood on the physical grounds as well. Indeed, the value of $g$ tends to 1 for highly extended in the forward direction single scattering diagrams. This makes the escape of photons more difficult. Actually, both their travel distance and travel time increase before their escape from a turbid medium as $g \to 1$. This, of course, will lead to the increase of light absorption by a medium. Clearly, the reflection function will decrease in this case.

It should be stressed that the reflection function $R_\infty^0(\xi, \eta, \varphi)$ of a semi-infinite nonabsorbing medium only weakly depends on its microstructure (9). Thus, reflection functions of absorbing semi-infinite turbid media with the same values of $y$ (but, possibly, with different values of $\beta$ and $g$) do not differ very much (see Eq. [l]). On the other hand, these turbid media could be quite different as far as their microstructure (size and shape of particles) is concerned. This simplifies the solution of the spectroscopic problem in a great extent.

To check the accuracy of Eq. [l], we plot the results of numerical calculations of the function

$$F = \frac{1}{U(1, 0.65, 0)} \ln \frac{R_\infty^0(1, 0.65, 0)}{R_\infty(1, 0.65, 0)} \qquad [7]$$

against $y$ in Fig. 2 for cloudy media of different microstructures. The numerical code, developed by Mishchenko *et al.* (8), was used to find values of $R_\infty^0(1, 0.65, 0)$ and $R_\infty(1, 0.65, 0)$. The function $U(1, 0.65, 0)$ was calculated with Eqs. [3] and [6]. Clearly, the relation [1] holds till $y \geq 0.5$. For larger $y$, however, symbols in Fig. 2 start to depart from the straight line $F = y$. This is especially pronounced at $y > 1$. Such a behavior is due to assumptions used in the derivation of Eq. [l], which is valid only for weakly absorbing media (4). Note that symbols for different sizes of particles at $y > 1$ still lay on the same line. This makes it possible to suggest that the function $R_\infty(\xi, \eta, \varphi)$ still depends primarily on the parameter $y$ (and not separately on parameters $\omega_0$ and $g$ or other characteristics of the phase function of the turbid medium under study). However, this dependence is more complicated than that presented in Eq. [1]. One obtains by fitting data in Fig. 2,

$$F = y(1 - cy), \qquad [8]$$

where $c = 0.05$.

The accuracy of this fitting procedure is quite high as it follows from Fig. 3. Thus, we have instead of Eq. [1]:

$$R_\infty(\xi, \eta, \varphi) = R_\infty^0(\xi, \eta, \varphi) \exp(-y(1 - cy)U(\xi, \eta, y)). \qquad [9]$$

This equation can be used to improve the accuracy of Eq. [1] at $y > 0.5$. Note that the convenient and accurate analytical

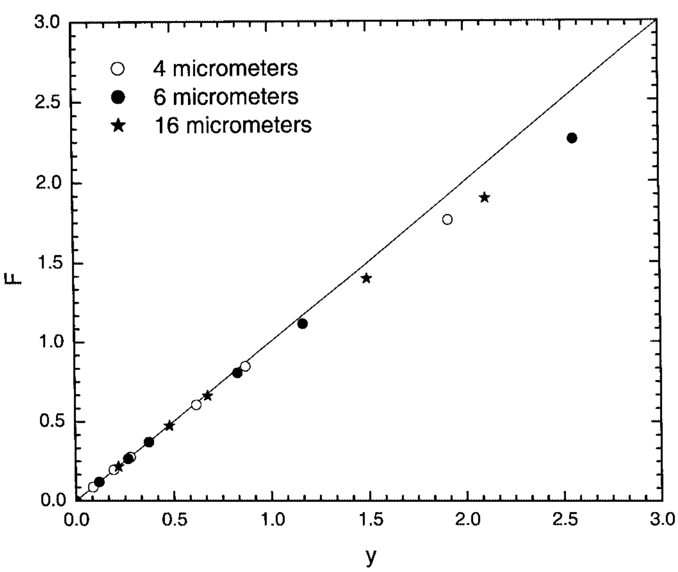


**FIG. 2.** The dependence $F(y)$ (see Eq. [7]) at $a_{ef} = 4$, 6, and 16 $\mu$m. Other input parameters are the same as in Fig. 1.

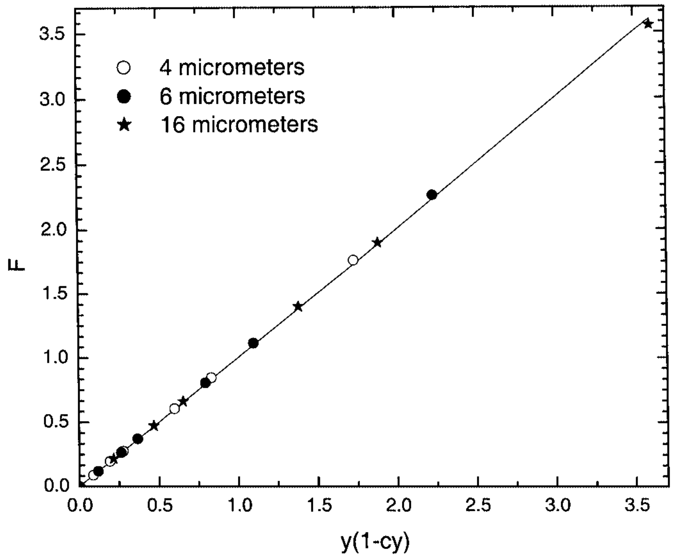


**FIG. 3.** The same as in Fig. 2 but for the dependence $F(y(1-cy))$.

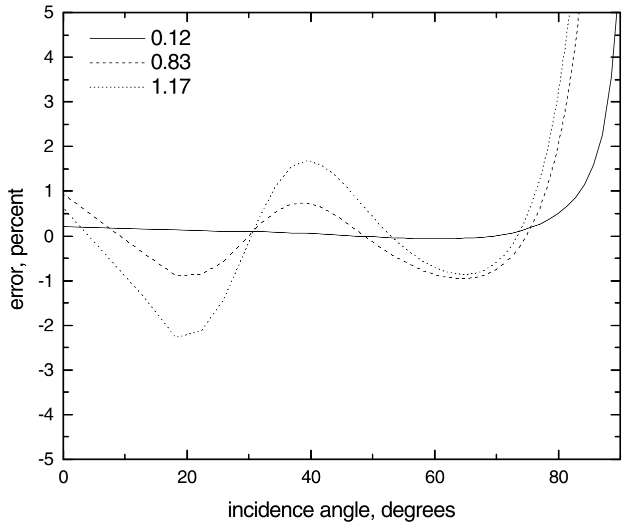


**FIG. 4.** The error of Eq. [9] at $y = 0.12(\chi = 0.00001)$, $0.83(\chi = 0.005)$, $1.17(\chi = 0.001)$ as function of the incidence angle at nadir observation. Other input parameters are as in Fig. 1.

expression for the reflection function $R^0_\infty(\xi, \eta, \varphi)$ was given by Kokhanovsky (9).

Equation [9] can be used to find the function $y(\lambda)$ from spectral measurements of the reflection function $R_\infty(\lambda)$. Namely, we have

$$y = \frac{1 - \sqrt{1 - 4\Phi c}}{2c}, \qquad [10]$$

where

$$\Phi = \frac{1}{U} \ln\left(\frac{R^0_\infty}{R_\infty}\right). \qquad [11]$$

Clearly, the function $y(\lambda)$ is related to the absorption coefficient $\chi(\lambda)$ of substances inside small particles. For instance, we have for large compared to the wavelength weakly absorbing particles (6),

$$\beta = B\chi(\lambda)x_{ef}, \qquad [12]$$

where $x_{ef} = 2\pi a_{ef}/\lambda$. The value of $B$ does not depend on the absorption coefficient and size of particles. It can depend, however, on their shape and the real part of the refractive index (6). Thus, we obtain with account for Eq. [2]: $\chi(\lambda) \sim y^2(\lambda)$.

Note that the value of $c$ was obtained by fitting exact radiative transfer calculations at a fixed observation geometry ($\xi = 1$, $\eta = 0.65$). Performing the same fitting procedure for other values of $\eta$ at $\xi = 1$, we found that the following simple approximate equation holds:

$$c(\eta) = 0.05 - 0.19\eta + 0.26\eta^2. \qquad [13]$$

Due to the reciprocity principle (1), the same equation is valid for the function $c(\xi)$ at $\eta = 1$. The error of approximation, given by Eq. [9] with account for Eq. [13], is presented in Fig. 4 at various values of $y$. We see that the error is smaller than 3% for incidence angles smaller than 80° and nadir observation. Thus, Eq. [9] can be safely used at $y < 1.17$. For larger $y$, however, the error increases. This is illustrated in Fig. 5. We see that the error can reach −30% at $y = 211.$ and $\vartheta_0 = 12°$. Remarkably, there is an angular region ($\vartheta_0 = 50$–$60°$), where the error of Eq. [9] is smaller than 7% even at $y = 2.6$ (see Fig. 5). This finding could be of importance for planning correspondent light scattering experiments. It follows from Fig. 5 that errors of Eq. [l] are larger than those of Eq. [9]. Thus, Eq. [9] should be preferably

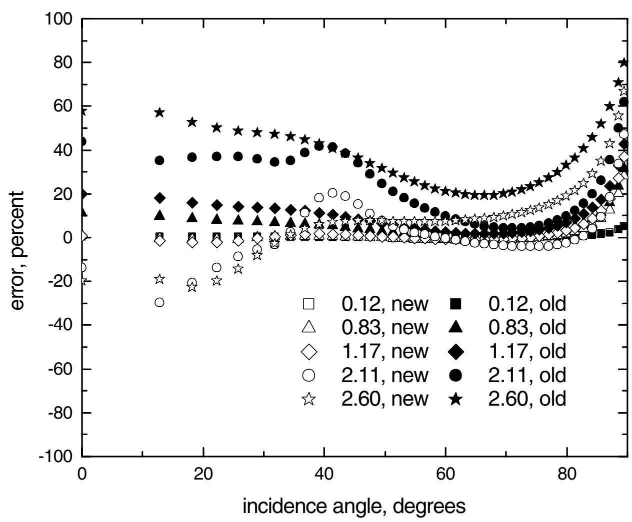


**FIG. 5.** The same as in Fig. 4, but also for new (Eq. [9]) and old (Eq. [l]) approximations at $y = 0.12(\chi = 0.00001)$, $0.83(\chi = 0.005)$, $1.17(\chi = 0.001)$, $y = 2.11(\chi = 0.001$, $a_{ef} = 16\ \mu$m), $y = 2.6(\chi = 0.005)$, as the function of the incidence angle at nadir observation.

used for the interpretation of experimental data. Note that Eq. [9] transforms to more simple Eq. [1] as $y \to 0$. Thus, both formulae have comparable accuracies at small values of $y$.

## RELATIONSHIP BETWEEN THE DEGREE OF POLARIZATION AND THE DIFFUSE REFLECTANCE

The aim of this section is to show how formulae presented above can be used for the interpretation of experimental results in light scattering media optics. For this, we take one example from a current research. Namely, Vitkin and Studinski (10) showed experimentally that the product of the degree of polarization in the backward direction $p$ to the diffuse reflection coefficient $r$ of selected human tissues of a different pigmentation is almost constant ($p \sim 1/r$ with the correlation coefficient 0.72). Thus, we have

$$pr = D, \qquad [14]$$

where $D \approx$ const. They have used the nadir illumination of skin samples, which are effectively semi-infinite light scattering media.

Let us show how Eq. [14] can be obtained from results presented above. For this we need to have the expression for the value of $r$, which is defined as follows (1):

$$r(\xi) = \frac{1}{\pi}\int_0^{2\pi} d\varphi \int_0^1 \eta\, d\eta\, R_\infty(\xi, \eta, \varphi). \qquad [15]$$

The approximate solution for $r(\xi)$ can be obtained expanding the exponent in Eq. [9] with respect to $y$ and performing analytical integration. The answer is

$$r(\xi) = \exp(-y(1 - cy)K_0(\xi)). \qquad [16]$$

Note that the value of $a(\xi) = 1 - r(\xi)$ gives us the portion of energy absorbed in the medium under its illumination along the angle arccos $\xi$. Taking the integral over incidence angle and following the same procedure as above, we obtain for the diffuse reflectance coefficient $\rho$ under diffuse illumination: $\rho = \exp(-y(1 - cy))$ approximately. The total absorption is proportional to $1 - \rho$ in this case. The ratio $(1 - \rho)/(1 - \omega_0)$ can be interpreted as an average number of scattering events in the medium.

Next we need the solution for the degree of polarization $p$. It was obtained by Kokhanovsky (11) as $y \to 0$,

$$p(\xi, \eta, \varphi) = p^0_\infty(\xi, \eta, \varphi)\exp(yU(\xi, \eta, \varphi)), \qquad [17]$$

where $p^0_\infty(\xi, \eta, \varphi) \equiv p(\xi, \eta, \varphi, \beta = 0)$. This equation was derived, using the Rozenberg formula [1]. Better accuracy for larger $y$ may be obtained if one applies Eq. [9]. Namely, using the same approach as discussed by Kokhanovsky (11), we arrive to the following approximate solution:

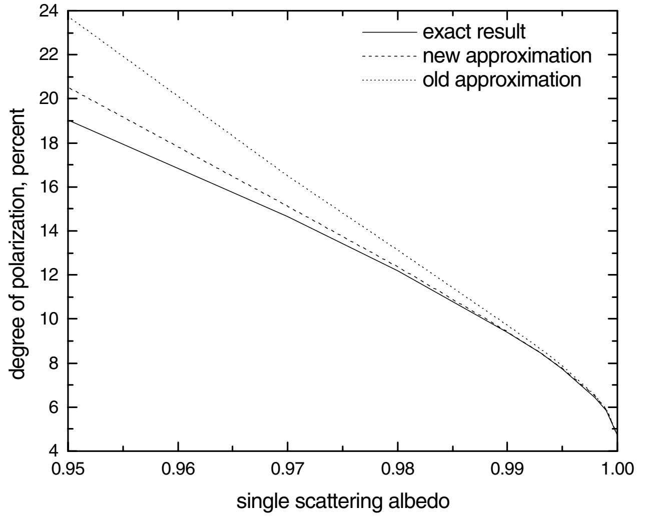


**FIG. 6.** The degree of polarization as the function of the single scattering albedo according to exact radiative transfer calculations, approximations [17] (old) and [18] (new). Input parameters were $\vartheta_0 = 6.7°$, $\vartheta = 32.5°$, $\varphi = 0°$, $g = 0.85$, $\omega_0 = 0.95, 0.97, 0.98, 0.99, 0.995, 0.998, 0.999$ for Deirmendjian's Cloud C1 phase function (12) at the wavelength 0.65 $\mu$m.

$$p(\xi, \eta, \varphi) = p^0_\infty(\xi, \eta, \varphi)\exp(y(1 - cy)U(\xi, \eta, \varphi)). \qquad [18]$$

We contrast results, derived from Eqs. [17] and [18] and the exact radiative transfer calculations in Fig. 6. We see that the use of Eq. [18] is preferable for larger values of $y$.

Now, taking the product of $r(1)$ and $p(1, 1)$, we arrive at Eq. [14] with

$$D = p^0_\infty(1, 1)\exp(\Xi), \qquad [19]$$

where

$$\Xi = \alpha y(1 - cy) \qquad [20]$$

and

$$\alpha = \frac{K_0^2(1)}{R^0_\infty(1, 1)} - K_0(1). \qquad [21]$$

We used values of $\xi = \eta = 1$ as in the experiment of Vitkin and Studinski (10) here. It follows that the value of $D$ is not constant due to the variability of parameters $p^0_\infty(1, 1)$, $\alpha$, and $y$ from one skin sample to another. This explains the scattering of experimental results around the average curve $p(1, 1) = \text{const}/r(1)$. The scattering is rather small. Therefore, we conclude that $\Xi$ is a small number and

$$D = p^0_\infty(1, 1). \qquad [22]$$

Our estimations give us that the value of $y$ varied in the range of 0.7–1.8 (except one point, where $r \approx 0.1$) in the experiment

of Vitkin and Studinski (10). We derived this result from their Fig. 4, using Eq. [9]. Also we estimated $\alpha$ as close to 0.17 for their experiment, where we used Eq. [6] at $\xi = 1$ and the value $R^0_\infty(1, 1) = 1.134$, given by Yanovitskii (13) for the Heney-Greenstein phase function at $g = 0.9$. This value of $g$ is typical for various tissues. Thus, we have $\Xi \approx 0.1$–$0.3$ and $\exp(\Xi) \approx 1.1$–$1.3$ with the average value of 1.2 over the whole interval of measurements.

Finally, we note that the relation

$$r(1)\, p(1, 1) = p^0_\infty(1, 1), \quad [23]$$

which is valid as $y \to 0$, can be easily checked experimentally.

## CONCLUSION

In conclusion, we have studied here the accuracy of Eqs. [1] and [9] for the calculation of reflection functions of semi-infinite turbid media. It was found that Eq. [9] improves the accuracy of Eq. [1] at $y > 0.5$. We also found that the parameter $c$ in Eq. [9] depends on the observation geometry. This dependence is given by Eq. [13] at nadir illumination of a scattering layer. Derived Eqs. [9] and [18] can be used for the interpretation of experiments in light scattering media optics as it was shown, applying them to experimental data of Vitkin and Studinski (11).

The final remark is that the dependence $p \sim 1/r$ is not a new result. It was etablished for the first time by Umow (13) and studied in detail by Ivanov (15, 16). Its application to the planetary optics is reviewed by Hapke (3).

The decrease of polarization with the reflectance of the turbid medium can be easely explained on physical grounds. Indeed, the high brightness of a turbid medium in reflected light is due to highly developed multiple light scattering, which randomizes both photon directions and polarization states.

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## REFERENCES

1. van de Hulst, H. C., "Multiple Light Scattering." Academic Press, New York, 1980.
2. Dubova, G. S., Khairullina, A. Ya., and Shumilina, S. F., *J. Appl. Spektr.* **27,** 871 (1997).
3. Hapke, B., "Theory of Reflectance and Emittance Spectroscopy." Cambridge, Univ. Press, Cambridge, UK, 1993.
4. Rozenberg, G. V., *Dokl. AN SSSR* **145,** 775 (1962).
5. Zege *et al.*, "Image Transfer through a Scattering Medium." Springer Verlag, New York, 1991.
6. Kokhanovsky, A. A., "Light Scattering Media Optics: Problems and Solutions." Springer-Verlag, Chichester, 2001.
7. Sobolev, V. V., "Light Scattering in Planetary Atmospheres." Nauka, Moscow, 1972.
8. Mishchenko, M. I., *et al.*, *J. Quant. Spectr. Rad. Transfer* **63,** 409 (1999).
9. Kokhanovsky, A. A., *J. Opt. Soc. Am.* **A19** (2002), in press.
10. Vitkin, I. A., and Studinski, R. C. N., *Opt. Commun.* **190,** 37 (2001).
11. Kokhanovsky, A. A., *J. Opt. Soc. Am.* **A18,** 883 (2001).
12. Deirmendjian, D., "Electromagnetic Scattering on Spherical Polydispersions." Elsevier, Amsterdam, 1969.
13. Yanovitskij, E. G., "Light Scattering in Inhomogeneous Atmospheres." Springer-Verlag, New York, 1997.
14. Umov, N., *Phys. Z.* **6,** 674 (1905).
15. Ivanov, A. P., *J. Techn. Phys.* **26,** 623 (1956).
16. Ivanov, A. P., *J. Techn. Phys.* **26,** 631 (1956).